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On Christoffel Type Functions for L_m Extremal Polynomials, I[1]

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The Christoffel functions for orthogonal polynomials are extended to the case of L_m extremal polynomials with an even integer m and their properties are given.

1. INTRODUCTION

Throughout of this paper let m be an even integer and $\mathbf{M} := \{0, 2, 4, ..., m-2\}$. Let α be a nondecreasing function on $\mathbb{R}$ with infinitely many points of increase such that all moments of $d\alpha$ are finite. The support of $d\alpha$ is the set of points of increase of $\alpha(x)$ and is denoted by $\operatorname{supp}(d\alpha)$. The smallest closed interval containing $\operatorname{supp}(d\alpha)$ is denoted by $\Delta(d\alpha)$. We call $d\alpha$ a measure. For $N \in \mathbb{N}$ let $\mathbf{P}_N$ denote the set of polynomials of degree at most N and $\mathbf{P}_N^*$ the subset of polynomials in $\mathbf{P}_N$ having real zeros only. We agree $\mathbf{P}_0^* = \mathbf{P}_0$. Put $\mathbf{P}_N^*(x) := \{P \in \mathbf{P}_N^* : P(x) = 1\}$ for $x \in \mathbb{R}$. ∂P stands for the exact degree of a polynomial P, i.e., $P \in \mathbf{P}_{\partial P} \backslash \mathbf{P}_{\partial P - 1}$. We define the L_m monic extremal polynomials

$$P_n(d\alpha, m; t) = t^n + \cdots, \qquad n = 0, 1, ...,$$

satisfying

$$\int_{\mathbb{R}} P_n(d\alpha, m; t)^m \, d\alpha(t) = \min_{P(x) = t^a + \cdots} \int_{\mathbb{R}} P(t)^m \, d\alpha(t). \tag{1.1}$$

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According to Theorem 4 in [1], if

$$x_{1n} > x_{2n} > \cdots > x_{nn} \tag{1.2}$$

are the zeros of $P_n(d\alpha, m; t)$ then the Gaussian quadrature formula with certain numbers $\lambda_{ik} := \lambda_{iknm}(d\alpha)$ (called the Cotes numbers of higher order)

$$\int_{\mathbb{R}} f(t)\, d\alpha(t) = \sum_{i=0}^{m-2} \sum_{k=1}^{n} \lambda_{ik} f^{(i)}(x_k) \tag{1.3}$$

is exact for all $f \in \mathbf{P}_{mn-1}$.

As we know, the case when $m=2$ is the special case of orthogonal polynomial; it has a long history of study and a classical theory. One of the important contents of this theory is the Christoffel functions

$$\lambda_n(d\alpha, x) := \min_{P \in \mathbf{P}_{n-1},\, P(x)=1} \int_{\mathbb{R}} P(t)^2\, d\alpha(t), \tag{1.4}$$

which are closely related to the Cotes numbers

$$\lambda_{0kn2}(d\alpha) = \lambda_n(d\alpha, x_{kn}), \qquad k = 1, 2, ..., n. \tag{1.5}$$

The study and applications of the Christoffel functions can be found in [2]. In this paper we will extend the Christoffel functions to the case of the L_m extremal polynomials and investigate their properties. Further investigations and applications will be given in forthcoming papers.

2. DEFINITION AND PROPERTIES

Given a fixed $x \in \mathbb{R}$ and $n \in \mathbb{N}$, for $P \in \mathbf{P}_{n-1}$ with $P(x)=1$ and $j \in \mathbf{M}$ let

$$A_j(P, x; t) := A_{jnm}(P, x; t)$$

$$:= \frac{1}{j!}(t-x)^j B_j(P, x; t)\, P(t)^m, \qquad B_j(P, x; \cdot) \in \mathbf{P}_{m-j-2}, \tag{2.1}$$

satisfy the conditions

$$A_j^{(i)}(P, x; x) = \delta_{ij}, \qquad i = 0, 1, ..., m-2. \tag{2.2}$$

It is easy to see that $A_j(P, x; t)$ must exist and be unique.

DEFINITION 1. The Christoffel type function $\lambda_{jnm}(d\alpha, x)$ $(j \in \mathbf{M})$ with respect to $d\alpha$ is defined by

$$\lambda_{jnm}(d\alpha, x) = \inf_{P \in \mathbf{P}^*_{a-1}(x)} \int_{\mathbb{R}} A_j(P, x; t)\, d\alpha(t). \tag{2.3}$$

Remark 1. For $n = 1$ it is easy to see that $\mathbf{P}^*_0(x) = \{1\}$, $A_j(P, x; t) = (t - x)^j / j!$, and

$$\lambda_{j1m}(d\alpha, x) = \frac{1}{j!} \int_{\mathbb{R}} (t - x)^j\, d\alpha(t).$$

In what follows we always assume $n \geqslant 2$.

LEMMA 1. *We have*

$$B_j(P, x; t): \sum_{i=0}^{m-j-2} b_i (t - x)^i, \tag{2.4}$$

where

$$b_i = b_i(P, x) = \frac{1}{i!} [P(t)^{-m}]^{(i)}_{t=x}, \qquad i = 0, 1, \ldots \tag{2.5}$$

Moreover, for $P \in \mathbf{P}^*_{n-1}(x)$ *and* $j \in \mathbf{M}$

$$b_{m-j-2} > 0, \qquad B_j(P, x; t) > 0, \qquad t \in \mathbb{R}. \tag{2.6}$$

Proof. Apply (1.3) and (2.8) in [3]. ∎

Let

$$P_\lambda(t) = P(t) + \lambda (t - x)\, Q(t), \qquad P \in \mathbf{P}^*_{n-1}(x), \qquad Q \in \mathbf{P}_{n-2}, \tag{2.7}$$

and put $f(\lambda, t) = A_j(P_\lambda, x; t)$ and $g(\lambda, t) = B_j(P_\lambda, x; t)$.

LEMMA 2. *Let for a fixed* $x \in \mathbb{R}$ *and* $j \in \mathbf{M}$ *a polynomial* $P \in \mathbf{P}_{n-1}(x)$ *satisfy*

$$\int_{\mathbb{R}} A_j(P, x; t)\, d\alpha(t) = \lambda_{jnm}(d\alpha, x). \tag{2.8}$$

If P_λ *in* (2.7) *satisfies*

> *Condition A*: *there is a number* $\delta > 0$ *such that* $P_\lambda \in \mathbf{P}^*_{n-1}(x)$ *holds for every* $\lambda \in [0, \delta]$,

then

$$\int_{\mathbb{R}} [(t - x)\, P(t)]^{m-1}\, q(t)\, d\alpha(t) \geqslant 0, \tag{2.9}$$

where

$$q(t)=(t-x)^{j-m+1}\,[\,g'_\lambda(0,t)\,P(t)+m(t-x)\,Q(t)\,g(0,t)] \tag{2.10}$$

and

$$q\in \mathbf{P}_{\max\{\partial P-1,\,\partial Q\}}\,. \tag{2.11}$$

Proof. By (2.8) and Condition A we conclude

$$\liminf_{\lambda\to+0}\int_{\mathbb{R}}\frac{f(\lambda,t)-f(0,t)}{\lambda}\,d\alpha(t)\geqslant 0.$$

Thus to prove (2.9) it suffices to establish

$$\liminf_{\lambda\to+0}\int_{\mathscr{R}}\frac{f(\lambda,t)-f(0,t)}{\lambda}\,d\alpha(t)=\int_{\mathbb{R}}f'_\lambda(0,t)\,d\alpha(t) \tag{2.12}$$

and

$$f'_\lambda(0,t)=\frac{1}{j!}\,[(t-x)\,P(t)]^{m-1}\,q(t). \tag{2.13}$$

To this end we observe that the function ($i\geqslant 0$)

$$d_i(P_\lambda,x)=\frac{1}{i!}\,[P_\lambda(t)^m]^{(i)}_{t=x}=\frac{1}{i!}\sum_{v=0}^{m}\binom{m}{v}\,[P(t)^{m-v}\,(t-x)^v\,Q(t)^v]^{(i)}_{t=x}\,\lambda^v$$

is a polynomial in λ of degree at most m. From the identity

$$[P_\lambda(t)^m\,P_\lambda(t)^{-m}]^{(i)}=0,\qquad i\geqslant 1,$$

applying the Newton–Leibniz formula and using (2.5) yields

$$\sum_{v=0}^{i}d_v(P_\lambda,x)\,b_{i-v}(P_\lambda,x)=0,\qquad i\geqslant 1.$$

Hence

$$b_0(P_\lambda,x)=1,\qquad b_i(P_\lambda,x)=-\sum_{v=1}^{i}d_v(P_\lambda,x)\,b_{i-v}(P_\lambda,x),\qquad i\geqslant 1.$$

Thus $b_i(P_\lambda, x)$ is also a polynomial in λ of degree at most im. By (2.1), (2.4), and (2.7) both $g'_\lambda(\lambda, t)$ and $f'_\lambda(\lambda, t)$ exist and are polynomials in λ of degree at most $m(m-j-1)$. Meanwhile

$$f'_\lambda(\lambda, t) = \frac{1}{j!}(t-x)^j \left[g'_\lambda(\lambda, t)\, P_\lambda(t)^m + m(t-x)\, Q(t)\, P_\lambda(t)^{m-1}\, g(\lambda, t) \right]. \tag{2.14}$$

Then the integral $\int_{\mathbb{R}} f'_\lambda(\lambda, t)\, d\alpha(t)$ converges uniformly for $\lambda \in [0, \delta]$. Hence (2.12) is true. On the other hand, by (2.14)

$$f'_\lambda(0, t) = \frac{1}{j!}(t-x)^j\, P(t)^{m-1} \left[g'_\lambda(0, t)\, P(t) + m(t-x)\, Q(t)\, g(0, t) \right]$$

is a polynomial in t. According to (2.2)

$$A_j^{(i)}(P_\lambda, x; x) - A_j^{(i)}(P, x; x) = \delta_{ij}, \qquad i = 0, 1, ..., m-2,$$

which shows that the polynomial $A_j(P_\lambda, x; t) - A_j(P, x; t) = f(\lambda, t) - f(0, t)$ in t contains the factor $(t-x)^{m-1}$, so does the polynomial $f'_\lambda(0, t)$. Thus (2.13) with (2.10) follows. Since $g'_\lambda(0, \cdot)$, $g(0, \cdot) \in \mathbf{P}_{m-j-2}$, (2.10) implies (2.11). ∎

LEMMA 3. *For a fixed* $x \in \mathbb{R}$ *and* $j \in \mathbf{M}$ *there exists a polynomial* $P \in \mathbf{P}^*_{n-1}(x)$ *such that* (2.8) *holds.*

Moreover, if (2.8) *is true then* $\partial P \geqslant n-2$, P *has distinct real zeros only, and*

$$\int_{\mathbb{R}} [(t-x)\, P(t)]^{m-1}\, q(t)\, d\alpha(t) = 0, \qquad \forall q \in \mathbf{P}_{n-2}. \tag{2.15}$$

Proof. Assume that $P_N \in \mathbf{P}^*_{n-1}(x)$ satisfies

$$\lim_{N \to \infty} \int_{\mathbb{R}} A_j(P_N, x; t)\, d\alpha(t) = \lambda_{jnm}(d\alpha, x).$$

Then

$$\int_{\mathbb{R}} A_j(P_N, x; t)\, d\alpha(t) \leqslant c < +\infty, \qquad \forall N \in \mathbb{N}.$$

Write

$$A_j(P_N, x; t) = \sum_{k=0}^{mn-2} a_{kN} t^k.$$

Thus the previous inequality implies by Theorem of Equivalent Norms of finite dimensional spaces that

$$|a_{kN}| \leqslant c_1 < +\infty, \qquad k = 0, 1, ..., mn-2, \qquad \forall N \in \mathbb{N}.$$

According to the Bolzano–Weierstrass Theorem by passing to a subsequence if necessary we may suppose that $P_N \to P$ $(N \to \infty)$. Then $P \in \mathbf{P}^*_{n-1}(x)$ and (2.8) holds.

Let us prove the second part of the lemma. Assume

$$P(t) = \prod_{k=1}^{p} \left(\frac{t-y_k}{x-y_k}\right)^{p_k},$$

where $+\infty > y_1 > y_2 > \cdots > y_r > -\infty$, $p_1, p_2, ..., p_r \in \mathbb{N}$.

Claim 1. $p_k = 1$, $k = 1, 2, ..., r$.

Suppose to the contrary that $p_k > 1$ for some k, $1 \leqslant k \leqslant r$. Choose $Q(t) = -(t-x)\, P(t)/(t-y_k)^2$, which obviously satisfies Condition A. In this case by (2.10) we can write

$$q(t) = C(t)\, Q(t), \tag{2.16}$$

where

$$C(t) = (t-x)^{j-m}\, [-g'_\lambda(0, t)(t-y_k)^2 + m(t-x)^2\, g(0, t)]$$

is a polynomial in t. Relation (2.11) shows $\partial q \leqslant \partial Q$ and hence $C(t) \equiv C$. By (2.6)

$$\begin{aligned} C = C(y_k) &= m(y_k - x)^{j-m+2}\, g(0, y_k) \\ &= m(y_k - x)^{j-m+2}\, B_j(P, x; y_k) > 0. \end{aligned}$$

Then

$$\int_{\mathbb{R}} [(t-x)\, P(t)]^{m-1}\, q(t)\, d\alpha(t) = -C \int_{\mathbb{R}} \frac{[(t-x)\, P(t)]^m}{(t-y_k)^2}\, d\alpha(t) < 0,$$

contradicting (2.9). This proves Claim 1.

Claim 2. $r \geqslant n-2$.

Suppose not and let $r < n-2$. By Claim 1 we have $\partial P = r < n-2$. Choose $Q(t) = -(t-x)\, P(t)$, which belongs to $\mathbf{P}_{n-2}$ and obviously satisfies Condition A. In the present case by (2.10) we get (2.16), where

$$C(t) = (t-x)^{j-m}\, [-g'_\lambda(0, t) + m(t-x)^2\, g(0, t)].$$

Again $C(t) \equiv C$. Since $\partial g'_\lambda(0, \cdot) < \partial[(\cdot - x)^2 g(0, \cdot)]$, by (2.6) $C = mb_{m-j-2} > 0$. This leads to a contradiction

$$\int_{\mathbb{R}} [(t-x)\, P(t)]^{m-1}\, q(t)\, d\alpha(t) = -C \int_{\mathbb{R}} [(t-x)\, P(t)]^m\, d\alpha(t) < 0$$

and proves Claim 2.

Claim 3. We have that

$$\int_{\mathbb{R}} \frac{(t-x)^{m-1}\, P(t)^m}{t-y_k}\, d\alpha(t) = 0, \qquad k = 1, 2, ..., r, \tag{2.17}$$

and if $r = n-2$ then

$$\int_{\mathbb{R}} (t-x)^{m-1}\, P(t)^m\, d\alpha(t) = 0. \tag{2.18}$$

To prove (2.17) choose $Q(t) = \pm P(t)/(t-y_k)$, $1 \leqslant k \leqslant r$, which obviously satisfies Condition A. By the same argument as above we obtain (2.16) and by (2.6)

$$\begin{aligned} C(t) &= (t-x)^{j-m+1} [\pm g'_\lambda(0, t)(t-y_k) + m(t-x)\, g(0, t)] \\ &\equiv C(y_k) = m(y_k - x)^{j-m+2}\, g(0, y_k) \\ &= m(y_k - x)^{j-m+2}\, B_j(P, x; y_k) > 0. \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}} [(t-x)\, P(t)]^{m-1}\, q(t)\, d\alpha(t) &= C(y_k) \int_{\mathbb{R}} [(t-x)\, P(t)]^{m-1}\, Q(t)\, d\alpha(t) \\ &\geqslant 0, \end{aligned} \tag{2.19}$$

which implies (2.17).

Similarly, choosing $Q(t) = \pm P(t)$ we can prove (2.18) if $r = n-2$.

Now we are in position to prove (2.15). If $r = n-1$ then (2.17) means (2.15), since the set $\{P(t)/(t-y_1), ..., P(t)/(t-y_{n-1})\}$ spans the space $\mathbf{P}_{n-2}$; if $r = n-2$ then (2.17) and (2.18) implies (2.15), since the set $\{P(t)/(t-y_1), ..., P(t)/(t-y_{n-2}), P(t)\}$ spans the space $\mathbf{P}_{n-2}$. ∎

The first main result in this paper is the following

THEOREM 1. *Let $x \in \mathbb{R}$ be fixed.*

(a) *There exists a unique polynomial $P \in \mathbf{P}^*_{n-1}(x)$ such that* (2.8) *holds for every $j \in \mathbf{M}$;*

(b) *$\partial P \geqslant n-2$ and P has distinct real zeros only;*

(c) *Equation* (2.8) *is true if and only if* (2.15) *holds;*

(d) *We have*

$$\lambda_{m-2,n,m}(d\alpha, x) = \min_{Q \in \mathbf{P}_{a-1},\, Q(x)=1} \frac{1}{(m-2)!} \int_{\mathbb{R}} Q(t)^m (t-x)^{m-2}\, d\alpha(t). \tag{2.20}$$

Proof. We distinguish the two cases when $j=m-2$ and $j<m-2$.

Case 1. $j=m-2$.

Let

$$G_x = \{(t-x)\, Q(t) \colon Q \in \mathbf{P}_{n-2}\}. \tag{2.21}$$

Let us consider the extremal problem: find $P \in \mathbf{P}_{n-1}$ such that $P(x)=1$ and

$$\int_{\mathbb{R}} P(t)^m (t-x)^{m-2}\, d\alpha(t) = \min_{Q \in \mathbf{P}_{n-1},\, Q(x)=1} \int_{\mathbb{R}} Q(t)^m (t-x)^{m-2}\, d\alpha(t). \tag{2.22}$$

It is easy to see that (2.22) is true if and only if $R = 1-P(\in G_x)$ satisfies

$$\int_{\mathbb{R}} [1-R(t)]^m (t-x)^{m-2}\, d\alpha(t) = \min_{Q \in G_x} \int_{\mathbb{R}} [1-Q(t)]^m (t-x)^{m-2}\, d\alpha(t). \tag{2.23}$$

But this is a problem of L_m approximation to the function 1 with respect to the measure $(t-x)^{m-2}\, d\alpha(t)$ from the $(n-1)$-dimensional subspace G_x. By [4, Corollary 2.2, p. 98, Corollary 3.5, p. 111, Theorem 1.11, p. 56] we conclude that there is a unique function $R \in G_x$ satisfying (2.23) and further (2.23) holds if and only if

$$\int_{\mathbb{R}} [1-R(t)]^{m-1} q(t)(t-x)^{m-2}\, d\alpha(t) = 0, \qquad \forall q \in G_x. \tag{2.24}$$

Recalling $R=1-P$, (2.24) is equivalent to (2.15). This means by (2.21) that there is a unique polynomial $P \in \mathbf{P}_{n-1}$ with $P(x)=1$ satisfying (2.22) and further (2.22) holds if and only if (2.15) is valid. Equation (2.15) shows that the polynomial $(t-x)\,P(t)$ in t changes sign at least $n-1$ times and hence $P(t)$ changes sign at least $n-2$ times. But $P \in \mathbf{P}_{n-1}$. So P has distinct real zeros only and hence $P \in \mathbf{P}^*_{n-1}(x)$. By (2.1), (2.4), and (2.5) we see

$$A_{m-2}(P, x; t) = \frac{1}{(m-2)!}\,(t-x)^{m-2}\,P(t)^m. \tag{2.25}$$

This proves Statements (a)–(d) for the case when $j=m-2$.

Case 2. $j<m-2$. In this case by Lemma 3 it suffices to show the uniqueness of a solution of (2.8) and the implication (2.15) ⇒ (2.8). To this end it is enough to establish the uniqueness of a solution satisfying (2.15), which is verified by Case 1. ▮

As a immediate consequence of Theorem 1 we state

Corollary 1. *We have*

$$\lambda_{0n2}(d\alpha, x) = \lambda_n(d\alpha, x). \tag{2.26}$$

Corollary 2. *If* $P \in \mathbf{P}^*_{n-1}(x)$ *satisfies* (2.8) *then the interval* $\Delta(d\alpha)$ *contains at least* $n-2$ *zeros of* P.

Proof. Suppose to the contrary that $\Delta(d\alpha)$ contains $r(\leqslant n-3)$ zeros of P, say, $y_1, ..., y_r$. For $q(t)=(t-x)(t-y_1)\cdots(t-y_r)$ we see that the polynomial $[(t-x)\,P(t)]^{m-1}\,q(t)$ does not change sign in $\Delta(d\alpha)$, which implies that its integral over $\Delta(d\alpha)$ is not zero, contradicting (2.15). ▮

The second main result in this paper is the following

Theorem 2. *We have*

$$\lambda_{jknm}(d\alpha) = \lambda_{jnm}(d\alpha, x_{kn}(d\alpha)), \qquad k=1, 2, ..., n, \qquad j \in \mathbf{M}. \tag{2.27}$$

Proof. Let k, $1 \leqslant k \leqslant n$, and $j \in \mathbf{M}$ be fixed. If (1.2) is the zeros of $P_n(d\alpha, m; t)$, then it follows from (1.1) by [4, Theorem 1.11, p. 56] that

$$\int_{\mathbb{R}} P_n(d\alpha, m; t)^{m-1}\,q(t)\,d\alpha(t) = 0, \qquad \forall q \in \mathbf{P}_{n-1},$$

or equivalently

$$\int_{\mathbb{R}} [(t-x_k)\, P(t)]^{m-1}\, q(t)\, d\alpha(t) = 0, \qquad \forall q \in \mathbf{P}_{n-1}, \tag{2.28}$$

where $P(t) = \prod_{i \neq k} [(t-x_i)/(x_k - x_i)]$. According to Theorem 1, Eq. (2.28) means

$$\lambda_{jnm}(d\alpha, x_k) = \int_{\mathbb{R}} A_j(P, x_k; t)\, d\alpha(t).$$

Inserting $f(t) = A_j(P, x_k; t)$ into (1.3) the above relation immediately gives (2.27). ▮

THEOREM 3. (a) *If* $d\alpha \leqslant d\beta$ *then*

$$\lambda_{jnm}(d\alpha, x) \leqslant \lambda_{jnm}(d\beta, x), \qquad x \in \mathbb{R}, \quad j \in \mathbf{M}; \tag{2.29}$$

(b) *we have*

$$\lambda_{0nm}(d\alpha, x) \geqslant \lambda_{mn/2}(d\alpha, x). \tag{2.30}$$

Proof. (a) Inequality (2.29) follows directly from (2.3).

(b) By (1.4)

$$\lambda_{mn/2}(d\alpha, x) = \min_{Q \in \mathbf{P}_{(mn/2)-1}} \frac{1}{Q(x)^2} \int_{\mathbb{R}} Q(t)^2\, d\alpha(t).$$

Then

$$Q(x)^2 \leqslant \lambda_{mn/2}(d\alpha, x)^{-1} \int_{\mathbb{R}} Q(t)^2\, d\alpha(t), \qquad Q \in \mathbf{P}_{(mn/2)-1}. \tag{2.31}$$

Let $P \in \mathbf{P}^*_{n-1}(x)$ satisfy (2.8) with $j = 0$. Since $A_0(P, x; t) \geqslant 0$ in $\mathbb{R}$, by [5, Theorem 1.21.2, p. 5] it may be written as

$$A_0(P, x; t) = R(t)^2 + Q(t)^2, \qquad R, Q \in \mathbf{P}_{(mn/2)-1}.$$

Thus by (2.31)

$$\begin{aligned} A_0(P, x; t) = R(t)^2 + Q(t)^2 &\leqslant \lambda_{mn/2}(d\alpha, t)^{-1} \int_{\mathbb{R}} [R(s)^2 + Q(s)^2]\, d\alpha(s) \\ &= \lambda_{mn/2}(d\alpha, t)^{-1} \int_{\mathbb{R}} A_0(P, x; s)\, d\alpha(s) \\ &= \lambda_{mn/2}(d\alpha, t)^{-1}\, \lambda_{0nm}(d\alpha, x). \end{aligned} \tag{2.32}$$

Putting $t=x$ we get

$$1 \leqslant \lambda_{mn/2}(d\alpha, x)^{-1}\, \lambda_{0nm}(d\alpha, x),$$

which is equivalent to (2.30). ∎

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